A Bochner Formula on Path Space for the Ricci Flow

Christopher Kennedy
christopher.kennedy@queensu.ca

Queen's University

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Overview of Results

- using stochastics, generalized classical Bochner formula for the heat flow on evolving manifolds $(M, g_t)_{t\in[0,T]}$ to infinite-dimensional Bochner formula for martingales on parabolic path space $P\mathcal{M}$ of space-time $\mathcal{M} = M \times [0, T]$
- characterize solutions of the Ricci flow in terms of Bochner inequalities on parabolic path space
- obtain gradient and Hessian estimates for martingales on $P\mathcal{M}$
- condensed proofs of prior characterizations of the Ricci flow

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Background on Characterizations

Probability in geometry. Used to better understand how shapes deform and evolve in time

Solutions to geometric flows. Two ways to characterize solutions include gradient estimates and Bochner formulas

Bochner inequality.

$$(-\partial_t + \Delta)|\nabla H_t f|^2 = 2|\nabla^2 H_t f|^2 + 2\mathrm{Rc}(\nabla H_t f, \nabla H_t f).$$

Example. Supersolutions to Einstein's equations $\mathrm{Rc} \geq 0$ satisfy equivalence

$$\mathrm{Rc} \geq 0 \iff (\partial_t - \Delta)|\nabla H_t f|^2 \leq -2|\nabla^2 H_t f|^2 \quad \iff |\nabla H_t f| \leq H_t |\nabla f|$$

Background on Characterizations

Evolving family of manifolds. Smooth and complete family of Riemannian manifolds $(M^n, g_t)_{t \in I}$ with bounded curvature and metric

Hamilton's Ricci flow. $\partial_t g_t = -2\mathrm{Rc}_{g_t}$ [Hamilton (1993)]

Solutions and supersolutions of flows. Can characterize solutions from following table

$\mathrm{Rc} \geq 0$	$\mathrm{Rc} = 0$
$\partial_t g_t \geq -2\mathrm{Rc}_{g_t}$	$\partial_t g_t = -2\mathrm{Rc}_{g_t}$

Background on Characterizations

Supersolutions. Equivalence of **supersolutions** to

$$\partial_t g_t \geq -2\mathrm{Rc}_{g_t}$$

(Ricci flow), classical Bochner inequality and gradient estimates

Ricci flow. Gradient estimate and the classical Bochner inequality for the Ricci flow are

$$\begin{cases} \partial_t g_t \geq -2\mathrm{Rc}_{g_t} & \iff |\nabla H_{st} f| \leq H_{st}|\nabla f| \\ & \iff (\partial_t - \Delta_{g_t})|\nabla H_{st} f|^2 \leq -2|\nabla^2 H_{st} f|^2 \end{cases}$$

- $f : M \to \mathbb{R}$ – test function
- $(M, g_t)_{t\in[0,T]}$ – Riemannian manifolds
- $H_{st} f$ – heat flow of $f : M \to \mathbb{R}$ starting at $f(\cdot, s)$

Solutions to Einstein's Equations

Solutions. Equivalence of **solutions** to $\mathrm{Rc} = 0$ to infinite-dimensional gradient estimate and infinite-dimensional Bochner inequality

Gradient estimate. Naber (2013) proved

$$\mathrm{Rc} = 0 \iff \left| \nabla_x \int_{PM} F \, d\mathbb{P}_x \right| \le \int_{PM} |\nabla_0^{||} F| \, d\mathbb{P}_x$$

Other work. Variants of estimate obtained [Cheng and Thalmaier (2018a), Cheng and Thalmaier (2018b), Wu (2020), Fang and Wu (2017), Wang and Wu (2018)]

Solutions to Einstein's Equations

Equivalence. Main result [Haslhofer and Naber (2018b)]

$$\mathrm{Rc} = 0 \iff d|\nabla_s^{||} F_t|^2 \geq \langle \nabla_t^{||} |\nabla_s^{||} F_t|^2, dW_t \rangle$$

Utility. Using Bochner, simplified proof of infinite-dimensional estimate

- $PM = C([0,\infty), M)$ – path space
- $F : PM \to \mathbb{R}$ – test function
- $F_t = \mathbb{E}[F \mid \Sigma_t]$ – induced martingale
- $\mathbb{P}_x$ – Wiener measure starting at $x \in M$
- $\nabla_t^{||}$ – parallel gradients

Solutions to the Ricci Flow

Main difference. Parabolic path space $P_T\mathcal{M}$ only consists of continuous space-time curves $\{\gamma_\tau = (T-\tau, x_\tau)\}$

Properties. Endowed with family of parabolic Wiener measures $\{\mathbb{P}_{(x,T)}\}$ and parabolic stochastic parallel gradients $\{\nabla^{||}_\sigma\}_{\sigma\geq 0}$

Result. Haslhofer and Naber (2018a) proved

$$\partial_t g_t = -2\mathrm{Rc}_{g_t} \iff \left|\nabla_x \int_{P_T\mathcal{M}} F \, d\mathbb{P}_{(x,T)}\right| \leq \int_{P_T\mathcal{M}} |\nabla^{||}_0 F| \, d\mathbb{P}_{(x,T)}$$

- $\mathcal{M} = M \times [0, T]$ – space-time equipped with connection
- $P_T\mathcal{M}$ – parabolic path space
- $\mathbb{P}_{(x,T)}$ – parabolic Wiener measures starting at $(x, T) \in \mathcal{M}$

Solutions to the Ricci Flow

Other work. Characterizations also by [Cheng and Thalmaier (2018b)], links between elliptic and parabolic setting [Cabezas-Rivas and Haslhofer (2020)]

Bochner inequality. Bochner formula on parabolic path space? [Kennedy (2023)]

Parabolic path space.

$$P_T\mathcal{M} := \left\{(x_\tau, T-\tau)_{\tau\in[0,T]} | x \in C([0,T], M)\right\}$$

Heat flow.

$$H_{st}f(x) = \int_M H(x,t|y,s)f(y)\, dV_{g_s}(y).$$

Solutions to the Ricci Flow

Example. $F(\gamma) = f(\pi_1 \gamma_{\tau_1})$ where $f : M \to \mathbb{R}$ and $\pi_1 : M \times [0, T] \to M$

Induced martingale.

$$F_\tau = \mathbb{E}[F|\Sigma_\tau]$$

given by

$$F_\tau(\gamma) = H_{T-\tau_1, T-\tau} f(\pi_1 \gamma_\tau)$$

Utility. Martingales generalize heat flow; motivates main result: a generalized Bochner formula on $P\mathcal{M}$

Geometric Preliminaries

Notations in time-independent geometry.

- M – complete Riemannian manifold
- $F_x = \{u : \mathbb{R}^n \to T_xM \, \text{orthonormal}\}$ – frames
- $\pi : F \to M$ – frame bundle
- $\tilde{f} = f \circ \pi : F \to \mathbb{R}$
- X^* – unique horizontal lift such that $\pi_*(X^*) = X$

Notations in time-dependent geometry.

- time-evolving family of manifolds: $(M, g_t)_{t\in[0,T]}$
- space-time with connection: $\mathcal{M} = M \times [0, T]$
- frames: $\mathcal{F}_{(x,t)} = \{u : \mathbb{R}^n \to (T_xM, g_t) \, \text{orthonormal}\}$
- frame bundle: $\pi : \mathcal{F} \to \mathcal{M}$
- $\mathcal{U} : P_0\mathbb{R}^n \to P_u\mathcal{F}$ – map
- $\Pi : P_u\mathcal{F} \to P_{(x,T)}\mathcal{M}$ – projection map

Geometric Preliminaries

Horizontal lift. Given vector $\alpha X + \beta\partial_t \in T_{(x,t)}\mathcal{M}$ and frame $u \in \mathcal{F}_{(x,t)}$, $\exists!$ horizontal lift $\alpha X^* + \beta D_t$ satisfying $\pi_*(\alpha X^* + \beta D_t) = \alpha X + \beta\partial_t$; X^* horizontal lift of $X \in T_x M$ with respect to fixed metric g_t

Local coordinates. $\mathcal{G}_{(x,t)} := \{u : \mathbb{R}^n \to (T_x M, g_t)\,\text{invertible, linear}\}$

Given local coordinates $(x^1, ..., x^n, t)$ on $\mathcal{M}$, get local coordinates (x^i, t, e_a^j) on $\mathcal{G}$, where $ue_a = e_a^j \frac{\partial}{\partial x^j}$

Canonical vector fields. [Hamilton (1993)]

$$\begin{cases} H_a & = e_a^j \frac{\partial}{\partial x^j} - e_a^j e_b^k \Gamma_{jk}^\ell \frac{\partial}{\partial e_b^\ell} \\ V_{ab} & = e_b^j \frac{\partial}{\partial e_a^j} - e_a^j \frac{\partial}{\partial e_b^j} \\ D_t & = \partial_t - \frac{1}{2}\widetilde{\partial_t g}_{ab} e_b^\ell \frac{\partial}{\partial e_a^\ell}, \end{cases}$$

Geometric Preliminaries

Derivatives of tensor fields. [Haslhofer and Naber (2018a)]

$$\begin{cases} \widetilde{\nabla_X T} = X^* \widetilde{T} \\ \widetilde{\nabla_t T} = D_t \widetilde{T} \\ \widetilde{\Delta T} = \sum_{a=1}^n H_a H_a \widetilde{T} =: \Delta_H \widetilde{T} \\ (\nabla^2 f)(ue_a, ue_b) = H_a H_b \widetilde{f} \end{cases}$$

Proposition. Let $\tilde{f} : \mathcal{F} \to \mathbb{R}$ be an orthonormally invariant function. Then

$$[D_t - \Delta_H, H_a]\tilde{f} = -\frac{1}{2}(\widetilde{\partial_t g} + 2\widetilde{\mathrm{Rc}})_{ab} H_b \tilde{f},$$

where $\widetilde{\mathrm{Rc}}_{ab}(u) = \mathrm{Rc}_{\pi(u)}(ue_a, ue_b)$.

Probabilistic Preliminaries

Horizontal curves. $\{u_\tau\}_{\tau\in[0,T]} \in \mathcal{F}$, where $\pi(u_\tau) = (x_\tau, T-\tau)$ correspond to curves $\{w_\tau\}_{\tau\in[0,T]} \in \mathbb{R}^n$ via IVP

$$\begin{cases} \frac{du_\tau}{d\tau} &= D_\tau + H_a(u_\tau)\frac{dw^a_\tau}{d\tau} \\ w_0 &= 0. \end{cases}$$

Anti-development in time-dependent setting. For evolving manifolds

$$\begin{cases} dU_\tau &= D_\tau\, d\tau + H_a(U_\tau) \circ dW^a_\tau \\ U_0 &= u. \end{cases}$$

Key probabilistic tool. [Haslhofer and Naber (2018a)] The stochastic differential equation

$$dU_\tau = D_\tau \, d\tau + H_a(U_\tau) \circ dW^a_\tau$$

has unique solution U_τ satisfying

$$d\tilde{f}(U_\tau) = D_\tau \tilde{f}(U_\tau)\, d\tau + \langle (H\tilde{f})(U_\tau), dW_\tau \rangle + \Delta_H(\tilde{f})(U_\tau)\, d\tau.$$

Idea of proof. Embed $\mathcal{F}$ into $\mathbb{R}^N$, extend and find solution up to explosion time $e(\mathcal{U})$ [Hsu (2002)]

Do some Itô calculus and project down from the frame to get the result!

Probabilistic Preliminaries

Brownian motion on space time. $\pi(U_\tau) = (X_\tau, T - \tau)$ Brownian motion on space time $\mathcal{M} = M \times I$ with base point $\pi(u) = (x, T)$

Stochastic parallel transport. Family of isometries

$$\left\{ P_\tau = U_0 U_\tau^{-1} : (T_{X_\tau} M, g_{T-\tau}) \to (T_x M, g_T) \right\}$$

Based path spaces.

$$P_u \mathcal{F} := \{ u_\tau | u_0 = u, \pi_2(u_\tau) = T - \tau \}_{\tau \in [0,T]} \subset \mathcal{F}$$

$$P_{(x,T)} \mathcal{M} = \{ \gamma_\tau = (x_\tau, T - \tau) | \gamma_0 = (x, T) \}_{\tau \in [0,T]},$$

Cylinder functions. $u \circ e_\sigma$, where $e_\sigma : P_{(x,T)}\mathcal{M} \to M^k$ are k-point evaluation maps, namely $e_\sigma : \gamma \mapsto (\pi_1\gamma_{\sigma_1}, ..., \pi_1\gamma_{\sigma_k})$, and $u : M^k \to \mathbb{R}$ is compactly supported

Stochastic parallel gradient. Defined by Fréchet derivative D_{V^σ}

$$D_{V^\sigma} F(\gamma) = \langle \nabla_\sigma^{||} F(\gamma), v \rangle_{(T_x M, g_T)},$$

for almost every Brownian curve γ and $v \in (T_x M, g_T)$, where $V_\tau^\sigma = P_\tau^{-1} v \chi_{[\sigma,T]}(\tau)$

Some more notations.

- $P_T\mathcal{M} := \left\{(x_\tau, T-\tau)_{\tau\in[0,T]} \middle| x \in C([0,T],M)\right\}$ – parabolic path space
- $H_{st}f(x) = \int_M H(x,t|y,s)f(y)\,dV_{g_s}(y)$ – solution to heat equation
- $\mathbb{P}_{(x,T)}[X_j \in U_j]$ – parabolic Wiener measures starting at $(x,T) \in M$
- $\mathbb{P}_u = U_*(\mathbb{P}_0)$, $\mathbb{P}_{(x,T)} = \Pi_*(\mathbb{P}_u)$ – induced Wiener measures
- $\nabla^{||}_\sigma F(\gamma) \in (T_xM, g_T)$ of $F : P_t\mathcal{M} \to \mathbb{R}$ – parabolic stochastic parallel gradient
- $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ – martingale on parabolic path space
- $F_\tau(\gamma) = \mathbb{E}[F|\Sigma_\tau](\gamma)$ – conditional expectation

Key example.

- $F(\gamma) = f(\pi_1 \gamma_{\tau_1})$ where $f : M \to \mathbb{R}$ and $\pi_1 : M \times [0, T] \to M$
- $F_\tau = \mathbb{E}[F \mid \Sigma_\tau]$ – induced martingale
- $F_\tau(\gamma) = \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma\big|_{[0,\tau]} * \gamma')\, d\mathbb{P}_{\gamma_\tau}(\tau')$ [Haslhofer and Naber (2018a)]

Calculation for $\tau > \tau_1$:

$$F_\tau(\gamma) = \int_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1 \gamma_{\tau_1})\, d\mathbb{P}_{\gamma_\tau}(\gamma') = f(X_{\tau_1})$$

Calculation for $\tau < \tau_1$:

$$\begin{aligned} F_\tau &= \int_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1 \gamma'_{\tau_1 - \tau})\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\ &= \int_M f(y) H(X_\tau, T-\tau | X_{\tau_1}, T-\tau_1)\, dV_{g_{T-\tau_1}}(y) \\ &= H_{T-\tau_1, T-\tau} f(X_\tau) \end{aligned}$$

Bochner Formula on Parabolic Path Space

Main result. Let $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be martingale on the parabolic path space of space-time. If $\sigma \geq 0$ is fixed, then

$$d(|\nabla^{||}_\sigma F_\tau|^2) = \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + (\dot{g} + 2\mathrm{Rc})_\tau(\nabla^{||}_\tau F_\tau, \nabla^{||}_\sigma F_\tau)\, d\tau$$
$$+ 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2 \delta_\sigma(\tau)\, d\tau.$$

Corollary. (Ricci $\implies$ Full Bochner) If $(M, g_t)_{t\in[0,T]}$ evolves under Ricci flow, then

$$d(|\nabla^{||}_\sigma F_\tau|^2) \geq \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau)$$

Bochner Formula on Parabolic Path Space

Steps to proof. Step #1: prove a martingale representation theorem: If $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ is a martingale on parabolic path space $P\mathcal{M}$, then it solves stochastic differential equation

$$\begin{cases} dF_\tau = \langle \nabla^{||}_\tau F_\tau, dW_\tau \rangle \\ F|_{\tau=0} = F_0. \end{cases}$$

- prove for k-point cylinder function ;
- lift $F_\tau : P\mathcal{M} \to \mathbb{R}$ to the frame bundle $\widetilde{F}_\tau : P\mathcal{F} \to \mathbb{R}$;
- show $d\widetilde{F}_\tau(\cdot) = \langle H^{(\ell+1)}(\widetilde{f})(\cdot), dW_\tau \rangle$;
- project back down

Bochner Formula on Parabolic Path Space

Steps to proof. Step #2: Must also find evolution of parallel gradient

$$d(\nabla^{||}_{\sigma} F_\tau) = \langle \nabla^{||}_{\tau} \nabla^{||}_{\sigma} F_\tau, dW_\tau \rangle + \frac{1}{2}(\dot{g} + 2\mathrm{Rc})_\tau (\nabla^{||}_{\tau} F_\tau)\, d\tau + \nabla^{||}_{\sigma} F_\sigma d\delta_\sigma(\tau)$$

Step #3: Lastly, a little Itô calculus to get the result!

New Characterizations of the Ricci Flow

Equivalence between Ricci and Bochner. For martingales $\{F_\tau\}$ on parabolic path space $P\mathcal{M}$, found new characterization of the Ricci flow via Bochner inequalities

Main theorem. (Full Bochner $\implies$ Ricci) If martingale $\{F_\tau\}$ satisfies the full Bochner inequality, then the family of manifolds $(M, g_t)_{t\in[0,T]}$ evolves under Ricci flow.

For martingales $\{F_\tau\}$ on parabolic path space $P\mathcal{M}$:

- new characterizations of the Ricci flow via Bochner inequalities
- gradient and Hessian estimates
- new proof of previous characterization of solutions of the Ricci flow [Haslhofer and Naber (2018a)]
- remark: newer work with generalized Ricci flow (M^n, g_t, H_t) by [Kopfer and Streets (2023)] includes torsion H

New Characterizations of the Ricci Flow

Theorem. (Full Bochner $\implies$ Weak Bochner)

$$d(|\nabla^{||}_\sigma F_\tau|^2) \geq \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2 \delta_\sigma(\tau)\, d\tau$$

$$\implies d|\nabla^{||}_\sigma F_\tau|^2 \geq \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau)$$

(Weak Bochner $\implies$ Linear Bochner $\implies$ Submartingale)

$$d|\nabla^{||}_\sigma F_\tau|^2 \geq \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau)$$

$$d|\nabla^{||}_\sigma F_\tau| \geq \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|, dW_\tau\rangle + |\nabla^{||}_\sigma F_\sigma|\, d\delta_\sigma(\tau)$$

$$\implies |\nabla^{||}_\sigma F_\tau| \leq \mathbb{E}[|\nabla^{||}_\sigma F_\tau| \Sigma_\tau]$$

- $(M^n, g_t)_{t\in I}$ - evolving family of manifolds
- F_τ - martingale on parabolic path space $P\mathcal{M}$

Gradient Estimates for Martingales on Path Space

Theorem. (Submartingale $\Longrightarrow$ Gradient Estimate #1)

$$|\nabla_\sigma^{||} F_\tau| \leq \mathbb{E}[|\nabla_\sigma^{||} F_\tau| \Sigma_\tau] \implies \left|\nabla_x \mathbb{E}_{(x,T)}[F]\right| \leq \mathbb{E}_{(x,T)}[|\nabla_x F|]$$

(Gradient Estimate #1 $\Longrightarrow$ Gradient Estimate #2)

$$\left|\nabla_x \mathbb{E}_{(x,T)}[F]\right| \leq \mathbb{E}_{(x,T)}[|\nabla_x F|] \implies |\nabla_\sigma^{||} F_{\tau_1}|^2 \leq \mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||} F_{\tau_2}|^2 |\Sigma_{\tau_1}\right]$$

- $F \in L^2(P\mathcal{M})$
- $F_\tau = \mathbb{E}[F|\Sigma_\tau]$ - induced martingale

Theorem. (Hessian estimate)

$$\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_\sigma|^2\right] + 2\mathbb{E}_{(x,T)}\int_0^T \left[|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\right] d\tau \le \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right]$$

(Poincaré Hessian Estimate)

$$\mathbb{E}_{(x,T)}\left[\left(F - \mathbb{E}_{(x,T)}[F]\right)^2\right] + 2\int_0^T\int_0^T \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\right] d\sigma\, d\tau \le \int_0^T \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right] d\sigma$$

- $(M^n, g_t)_{t\in I}$ - evolving family of manifolds
- $F \in L^2(P\mathcal{M})$

Condensed Proofs of Prior Characterizations

Strategy. Use previous gradient estimates to prove equivalent notions of the Ricci flow

Some of the estimates. (Gradient Estimate #1 $\implies$ Another Gradient Estimate)

$$\left|\nabla_x \mathbb{E}_{(x,T)}[F]\right| \leq \mathbb{E}_{(x,T)}[|\nabla_x F|] \implies \left|\nabla_x \mathbb{E}_{(x,T)}[F]\right| \leq \mathbb{E}_{(x,T)}[|\nabla_0^{||} F|]$$

(Gradient Estimate #2 $\implies$ Quadratic Variation Estimate)

$$|\nabla_\sigma^{||} F_{\tau_1}|^2 \leq \mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||} F_{\tau_2}|^2 \middle| \Sigma_{\tau_1}\right]$$
$$\implies \mathbb{E}_{(x,T)}\left[\frac{d[F,F]_\tau}{d\tau}\right] \leq 2\mathbb{E}_{(x,T)}\left[|\nabla_\tau^{||} F|^2\right].$$

Condensed Proofs of Prior Characterizations

Characterization. (Full Bochner $\implies$ Weak Bochner)

$$d|\nabla^{||}_{\sigma} F_{\tau}|^2 \geq \langle \nabla_{\tau} |\nabla^{||}_{\sigma} F_{\tau}|^2, dW_{\tau} \rangle + 2|\nabla^{||}_{\tau} \nabla^{||}_{\sigma} F_{\tau}|^2 \, d\tau + 2|\nabla^{||}_{\sigma} F_{\sigma}|^2 d\delta_{\sigma}(\tau)$$
$$\implies d|\nabla^{||}_{\sigma} F_{\tau}|^2 \geq \langle \nabla_{\tau} |\nabla^{||}_{\sigma} F_{\tau}|^2, dW_{\tau} \rangle + 2|\nabla^{||}_{\sigma} F_{\sigma}|^2 d\delta_{\sigma}(\tau)$$

(Weak Bochner $\implies$ Ricci)

$$d|\nabla^{||}_{\sigma} F_{\tau}|^2 \geq \langle \nabla_{\tau} |\nabla^{||}_{\sigma} F_{\tau}|^2, dW_{\tau} \rangle + 2|\nabla^{||}_{\sigma} F_{\sigma}|^2 d\delta_{\sigma}(\tau) \implies \partial_t g_t = -2\mathrm{Rc}_{g_t}$$

Also, log-Sobolev and Poincaré equivalencies for the Ricci flow

Converse implications involve substituting one-point and two-point cylinder test functions of compact support

Summary of Results

- using stochastics, generalized classical Bochner formula for the heat flow on evolving manifolds $(M, g_t)_{t\in[0,T]}$ to infinite-dimensional Bochner formula for martingales on parabolic path space $P\mathcal{M}$ of space-time $\mathcal{M} = M \times [0, T]$
- characterize solutions of the Ricci flow in terms of Bochner inequalities on parabolic path space
- obtain gradient and Hessian estimates for martingales on $P\mathcal{M}$
- condensed proofs of prior characterizations of the Ricci flow

Thank you for your attention!

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